AEC Computing and Applied Mathematics Center

AEC RESEARCH AND DEVELOPMENT REPORT

TID-4500
14th Ed.

NYO-2537
PHYSICS

PROPAGATION OF MAGNETOHYDRODYNAMIC WAVES WITHOUT RADIAL ATTENUATION

by

Harold Grad

January 15, 1959

Institute of Mathematical Sciences

NEW YORK UNIVERSITY

NEW YORK, NEW YORK

AEC Computing and Applied Mathematics Center
Institute of Mathematical Sciences
New York University

TID-4500
14th Ed.

NYO-2537
PHYSICS

PROPAGATION OF MAGNETOHYDRODYNAMIC WAVES WITHOUT RADIAL ATTENUATION

by

Harold Grad

January 15, 1959

Contract No. AT(30-1)-1480

NYO-2537
PHYSICS AND
MATHEMATICS

ABSTRACT

The propagation of hydromagnetic waves is examined in a perfectly conducting compressible fluid. The medium is highly anisotropic and exhibits strange properties, many of which are hidden by Fourier analysis or by examination of plane waves. In a certain sense, it is possible to separate an arbitrary wave motion into three modes (part of the separation is always possible, the remainder only in certain limiting cases). However, in separating, one can lose the domain of dependence properties of the original hyperbolic system. One of the modes (in certain limiting cases, two modes) represents one-dimensional wave propagation along magnetic field lines without three-dimensional radial attenuation. Depending on the amount of dissipation present, such signals could propagate large distances.

Contents

PROPAGATION OF MAGNETOHYDRODYNAMIC WAVES WITHOUT RADIAL ATTENUATION[1]

1. Introduction

We shall be concerned with the following magneto-hydrodynamic formulation:

$$(1)\quad \begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \\ \rho\left(\dfrac{\partial u}{\partial t} + u\cdot\nabla u\right) + a^2\nabla\rho = \dfrac{1}{\mu}\operatorname{curl} B \times B \\ \dfrac{\partial B}{\partial t} = \operatorname{curl}(u \times B), \quad \operatorname{div} B = 0\ . \end{cases}$$

This system is essentially that of Lundquist[2] and is the simplest formulation of compressible magnetohydrodynamics. It follows from the removal of all dissipative mechanisms; more precisely, the fluid is isentropic with a scalar pressure, and Ohm's law is taken in the simple form $E + u \times B = 0$. Also, displacement current and electrostatic forces have been dropped; thus the system (1) is Galilean invariant and of conservation form.[3] We are using rationalized MKS units

---

[1] This report contains the text of an address presented at the Third Lockheed Symposium on Magnetohydrodynamics on November 21, 1958.
The author wishes to express his appreciation to Jack Bazer, K. O. Friedrichs, and Herbert Kranzer for valuable discussions on this subject.

[2] S. Lundquist, Studies in magneto-hydrodynamics, Arkiv für Fysik, Band 5, (1952).

[3] For a discussion of these points see A. A. Blank and Harold Grad, Fluid Magnetic Equations - General Properties, NYO-6486, Notes on Magneto-Hydrodynamics VI, Inst. of Math. Sci., N.Y.U., (July 1, 1958).

for B; ρ and u are the gas density and flow velocity vector, and $a^2 = \partial p/\partial \rho$ is the ordinary gas sound speed.

The theory of this nonlinear system has been shown to be basically similar to that of ordinary compressible gas dynamics. In particular, one-dimensional simple wave and shock wave solutions have been investigated.[1] Nevertheless, in detail, the magnetohydrodynamic phenomena are decidedly more complex. One distinction is that there are three characteristic (or propagation) speeds for the magnetohydrodynamic system in contrast with the single sound speed, a, of gas dynamics. What is probably more significant, however, is the strongly anisotropic nature of the electrically conducting gas. In fact, two of the three propagation speeds reduce to zero in a direction perpendicular to the magnetic field. This is sufficiently degenerate to render inapplicable most of the conventional theory of wave propagation in anisotropic media.

Most of the analysis will refer to the linearization of (1):

---

1 K. O. Friedrichs and H. Kranzer, Nonlinear Wave Motion, NYO-6486, Notes on Magneto-Hydrodynamics VIII, Inst. of Math. Sci., N.Y.U., (July 31, 1958).

Jack Bazer, Resolution of an Initial Shear Flow Discontinuity in One-Dimensional-Hydromagnetic Flow, Research Report No. MH-5, Div. of Electromagnetic Research, Inst. of Math. Sci., N.Y.U. (June, 1956).

W. B. Ericson and Jack Bazer, Hydromagnetic Shocks, Research Report No. MH-8, Div. of Electromagnetic Research, Inst. of Math. Sci., N.Y.U. (January, 1958).

$$(2) \quad \begin{cases} \frac{\partial \rho}{\partial t} + \rho_o \operatorname{div} u = 0 \\ \rho_o \frac{\partial u}{\partial t} + a_o^2 \nabla \rho = \frac{1}{\mu} \operatorname{curl} B \times B_o \\ \frac{\partial B}{\partial t} = \operatorname{curl} (u \times B_o), \quad \operatorname{div} B = 0 \ . \end{cases}$$

The unperturbed quantities are denoted by ρ_o, B_o, a_o, $u_o = 0$, while ρ, u, B here denote the perturbations; B_o represents a uniform magnetic field.

Since this system of equations is hyperbolic, its solutions can be described in terms of the propagation of waves. As in the case of the ordinary wave equation, one can investigate a simplified problem, viz. the propagation of wave fronts and, associated with this, one can construct a theory of ray optics.[1] An important point to realize is that the first order (Hamilton-Jacobi) partial differential equation which governs the behaviour of wave fronts does not describe the entire content of the second order wave equation. In particular, there may be a complicated non-steady flow left behind a wave front propagating into still air, and this residue can only be analyzed by the use of the full wave equation. The situation is more complicated in the magnetohydrodynamic case where there are, in general, three distinct wave fronts which propagate at different speeds.

---

[1] Jack Bazer and O. Fleischman, Geometric Hydromagnetics, Research Report No. MH-12, Div. of Electromagnetic Research, Inst. Math. Sci., N.Y.U. (February 1959).

For example, a wave front which moves at one of the two slower speeds can leave a "residue" which propagates <u>ahead</u> of itself as well as behind.

This remark introduces the major points of investigation of this paper: first, the discovery of special solutions which propagate at a single one of the characteristic speeds, and second, the question of the resolution of an arbitrary solution of the linear system (2) into three "modes" which propagate at distinct speeds. In one dimension, a complete separation can be obtained using the Riemann invariants of the linear system (2). For the non-linear system, in one dimension, special <u>simple wave</u> solutions which propagate at a single speed can be found. Separation of the wave solutions of the linear system (2) can be only partially accomplished in more than one dimension, and the nature of this separation is quite subtle (section 7). However, it is easy to exhibit a large class of special solutions which propagate at a single one of the characteristic speeds (sections 5 and 6).

In contrast to the problem of the separation of general solutions, wave fronts can always be separated, and even in the case when the unperturbed magnetic field, B_o, is not uniform.[1]

---

[1] See footnote on page 6.

It should be remarked that a Fourier expansion of the linear system (2) immediately yields separable modes. This resolution is artificial, however, and hides some of the most important properties of the solutions. For example, a Fourier resolution does not easily answer the question whether one can create an initial distrubance which is confined to a finite region and which propagates as a single one of the three modes.[1] With modes defined in a more physically appealing sense than as Fourier components, the modes cannot be entirely decoupled even for the linearized system.

The fact that two of the propagation speeds are in some sense zero in a direction perpendicular to B_o suggests the possibility of waves which propagate one-dimensionally without radial spreading. However, the situation is complicated by the interaction between different modes (eg. a wave front of one type can leave a residue involving the other two modes) and by some ambiguity in the meaning of "zero propagation speed" perpendicular to B_o (associated with non-convexity of one of the characteristic cones). The analysis made here is a step in the direction of mathematical verification of certain qualitative explanations that have been offered of the phenomenon of "whistlers", but there is considerably more to be done in this direction (section 9).

---

[1]A Fourier analysis of the radiation from a point source has been made by Leo C. Levitt in his thesis, California Institute of Technology, 1957. However, some of his results are difficult to interpret because of domain of dependence ambiguities.

2. Characteristics

A characteristic surface $\phi(x,y,z,t) = \text{constant}$, (or wave front) of the system (1) can be defined as a surface in space-time across which there can exist discontinuities of some of the first derivatives of $\rho, u,$ or B. The system of equations (1) imposes certain relations among the relative magnitudes of the jumps in the various derivatives; specifically, it reduces to a system of seven (scalar) homogeneous linear equations in the jumps of the seven derivatives of ρ, u, B normal to the surface. Since these equations are homogeneous, solutions exist only when the determinant vanishes. This condition restricts the possible orientation which can be taken by the surface element $\phi = \text{constant}$ at a given point (x,y,z,t) and is expressed as a homogeneous polynomial in the derivatives $(\phi_x, \phi_y, \phi_z, \phi_t)$ which define the normal to a characteristic surface element. It is convenient to represent this normal as a vector $(n_1, n_2, n_3, -c)$ with unit projection on physical space, $n_1^2 + n_2^2 + n_3^2 = 1$. The characteristic equations are then[1]

$$
(3) \qquad \left\{
\begin{aligned}
(u_n - c)\delta\rho + \rho\delta u_n &= 0 \\
\rho(u_n - c)\delta u + a^2 n\delta\rho &= \frac{1}{\mu}(n \times \delta B) \times B \\
(u_n - c)\delta B - u\delta B_n - B_n\delta u + B\delta u_n &= 0, \quad \delta B_n = 0 \quad .
\end{aligned}
\right.
$$

We have written $\delta\rho$ for the jump in the normal derivative of

---

[1] Friedrichs and Kranzer, footnote on page 5.

ρ , etc. since one can show that the same characteristic relations hold for small jumps in ρ itself as for finite jumps in $\partial\rho/\partial n$.

It should be remarked that the system (1) is mathematically sound even when the restriction $\operatorname{div} B = 0$ is not imposed. A property of this extended system is that $\frac{\partial}{\partial t}(\operatorname{div} B) = 0$ for all time. This reduces the system (1) to sixth order, and the characteristic equations (3) to a set of six independent equations.

The determinant condition is a homogeneous polynomial of the sixth degree in $(n_1, n_2, n_3, -c)$:

$$(4) \quad [c^2 - B_n{}^2/\mu\rho][c^4 - (a^2 + B^2/\mu\rho)c^2 + a^2 B_n{}^2/\mu\rho] = 0, \; B_n = B\cdot n$$

The linear system (2) which is homogeneous in first derivatives represents a non-dispersive medium. It is easy to verify that this characteristic determinant gives, in the linear case, the speed of propagation, c , of a plane wave in the direction n_i .

It is convenient to represent these characteristic speeds as a polar plot; on a ray with direction n_i , we lay off radii equal to the corresponding values of c . We call this the normal speed locus. For the ordinary wave equation, it is a sphere centered at the origin. For the present equations, the result is as shown in Figure 1. There are two possibilities depending on whether the Alfven speed,

$$(5) \qquad A = (B^2/\mu\rho)^{1/2}$$

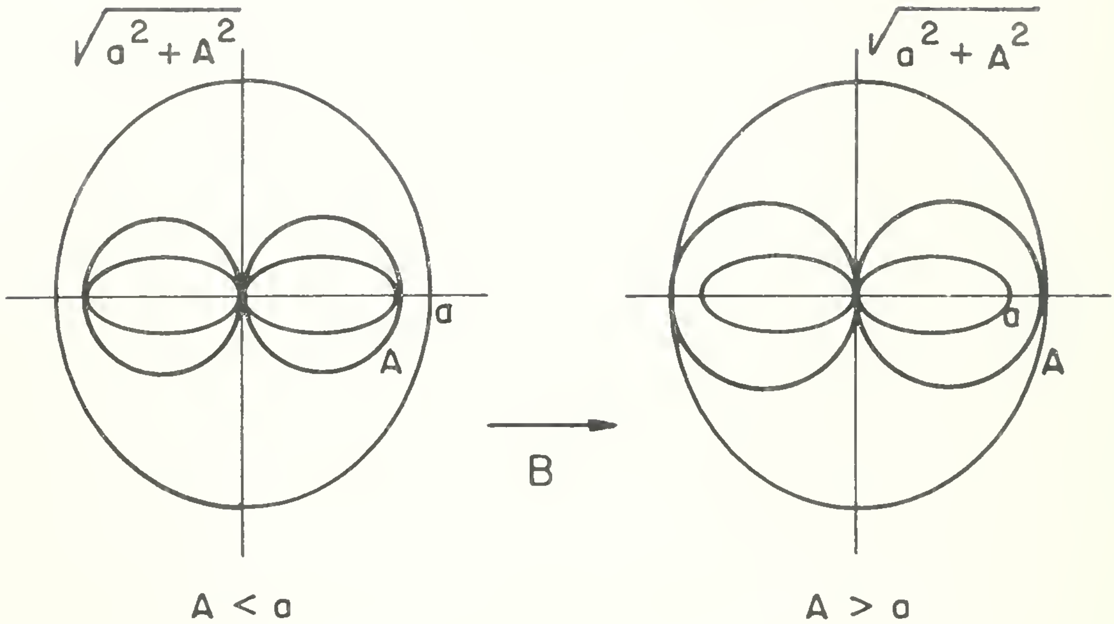

Figure 1

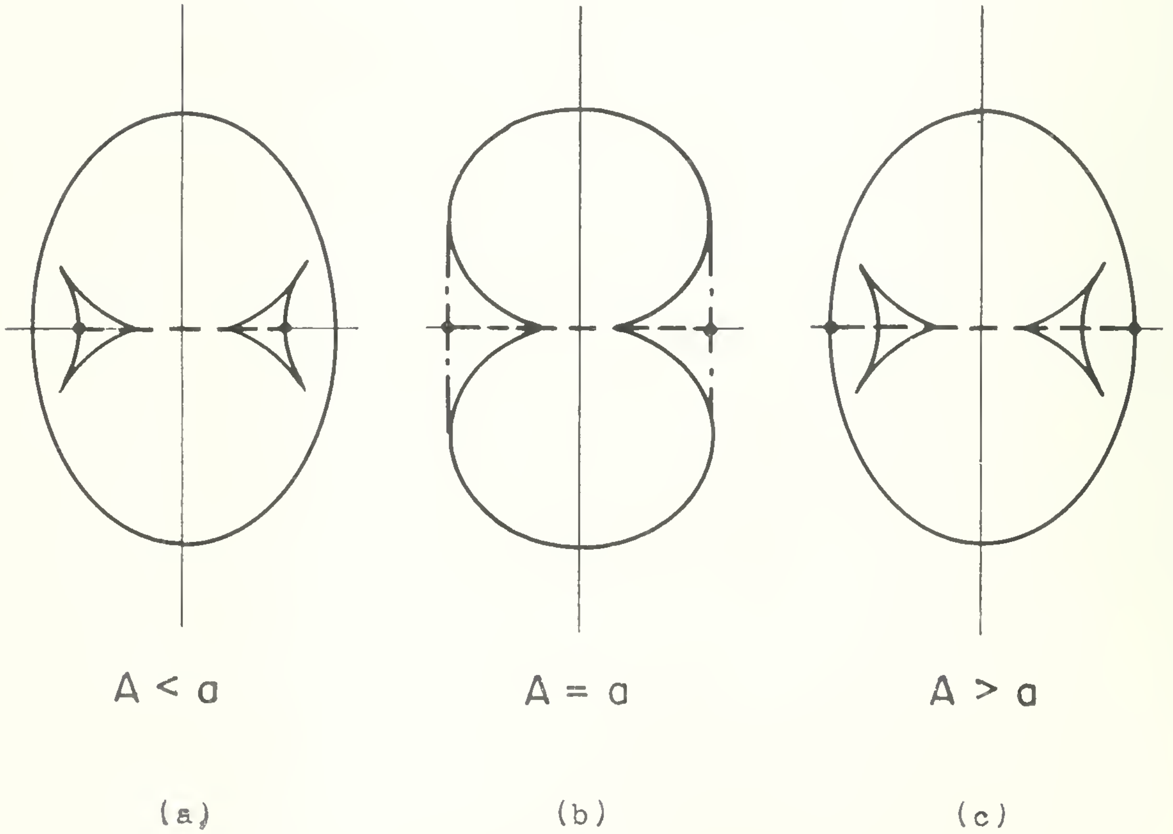

Figure 2

is larger or smaller than the gas sound speed, a . The transverse locus, which is given by the first factor of (4), consists of two spheres through the origin. The transverse locus separates the <u>slow</u> locus (which also has two lobes, meeting at the origin) from the <u>fast</u> locus which is a simple convex surface centered about the origin. In any given direction, n , a slow plane wave will propagate slower than the transverse wave in the same direction, and this, in turn, will be slower than the corresponding fast plane wave. For $A > a$, the transverse locus touches the fast locus when n is parallel to B, and for $A < a$, it touches the slow locus. It is evident that plane waves of either slow or transverse type propagate with zero speed when aimed perpendicular to B. Using this normal speed locus, it is possible to describe a simple construction (finite difference scheme) to obtain a characteristic surface, $\phi(x,y,z,t) = \text{constant}$, given its intersection with the initial manifold, $t=0$. At each point of the wave front, we construct the normal and lay off the length c dt where c corresponds (on the normal speed locus, Fig. 1) to the given direction of the normal. This construction merely expresses the definition of $(n_1, n_2, n_3, -c)$ as the four-dimensional normal to the characteristic surface. The construction is essentially correct, but as we shall soon see, it can break down after some time. It follows immediately from this construction that, if an initial wave front splits into slow, transverse, and fast

components, the slow front, at any later time, will not have progressed as far as the transverse wave. For, at the initial instant it is slower than the transverse wave in every direction, and it can only pass the transverse wave at some later time by touching it at some point, and at this presumed point of tangency its speed would again be less.

For many purposes, a more important locus is the one dual to the normal speed locus. Through each point of the normal speed locus we pass a plane normal to n; this represents a possible element of a characteristic surface $\phi =$ const. The <u>characteristic locus</u> (Fig. 2) is the envelope of these planes; ordinarily one speaks of the <u>characteristic cone</u> which is the "surface" in (x, y, z, t) obtained by passing a ray from the origin to each point of the characteristic locus considered to be drawn in the three-space $t = 1$; (the term "cone" should not be taken too literally, since the envelope can be quite degenerate).

More precisely, one considers an infinitesimal characteristic cone attached to each point (x, y, z, t), since the characteristic speeds are variable in general. A characteristic surface has the property that it is everywhere tangent to the local characteristic cone. The finite characteristic surface, $\phi =$ constant, is obtained by piecing together characteristic surface elements in two stages, first taking an envelope at a given point (x, y, z, t), then taking an envelope of these cones.

For the special case of linear equations with constant co-efficients (eg. the system (2)), the characteristic cone is independent of x,y,z, and t. Consequently, the evolution of a wave front from time t_o to time t_1 can be obtained in one step by attaching to each point of the initial wave front a characteristic locus whose size is given by the scale factor $t_1 - t_o$ and then taking an envelope. In particular, the characteristic locus itself can be thought of as the wave front which emerges from a point disturbance after a finite time.

3. Propagation of Wave Fronts

At a given point (x,y,z,t), the characteristic system (3) can be considered to be a set of linear equations with c as an eigenvalue parameter. To each eigenvalue c (cf. equation (4)), there exists a corresponding eigenvector $(\delta\rho, \delta u, \delta B)$. Expansion of an arbitrarily given set of jumps $(\delta\rho, \delta u, \delta B)$ into eigenvectors amounts to a resolution of the discontinuity into six modes which propagate at the various characteristic speeds.

Description of the propagation of the various wave fronts is facilitated by introducing the rays or bicharacteristics of the original system.[1] The equation for a characteristic surface or wave front is (cf. equation (4)):

$$(6) \quad [\phi_t^2-(A\cdot\nabla\phi)^2][\phi_t^4-(a^2+A^2)\phi_t^2(\nabla\phi)^2+a^2(\nabla\phi)^2(A\cdot\nabla\phi)^2] = 0 \ .$$

We have introduced the vector Alfven speed, $A = B/\sqrt{\mu\rho}$, for notational simplicity. This is a first order partial differential equation and itself has characteristics which, because of the special form of the equation, can be described as curves rather than as cones. For simplicity we consider (6) to be solved for $\phi_t = F(\nabla\phi, a, A)$ (six roots). The characteristics of (6)

$$(7) \quad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial F}{\partial \nabla\phi} \\[2ex] \dfrac{d\nabla\phi}{dt} = -\dfrac{\partial F}{\partial x} = -\left(\dfrac{\partial F}{\partial a}\dfrac{\partial a}{\partial x} + \dfrac{\partial F}{\partial A}\cdot\dfrac{\partial A}{\partial x}\right) \\[2ex] \dfrac{d\phi_t}{dt} = -\dfrac{\partial F}{\partial t} = -\left(\dfrac{\partial F}{\partial a}\dfrac{\partial a}{\partial t} + \dfrac{\partial F}{\partial A}\cdot\dfrac{\partial A}{\partial t}\right) \end{cases}$$

---

[1] A detailed analysis of this theory is given by Bazer and Fleischman, see footnote on page 6.

These equations yield not only the rays $x(t)$ but also the propagation along the rays of the characteristic surface element $\phi_t(t)$, $\nabla\phi(t)$. In the special linear case with constant coefficients (i.e. (2)), we see that the rays are straight lines and the surface element is carried parallel along a ray. In the characteristic locus, Fig. 2, the ray corresponding to a given surface element is the line from the origin to the point on the locus which has the given tangent direction. The image of a wave front at a finite later time can be constructed without taking envelopes merely by laying off at each point the ray of finite length (taken from Fig. 2) which corresponds to the direction of the wave front at the given point; in addition, we know that the tangent plane will be the same at the end of the ray. This construction should be compared with the previous one using normals to the front and the normal speed locus.

The most significant portion of the characteristic locus (Fig. 2) from the point of view of general theory is the fast locus since this determines the domain of dependence. This locus is a simple convex surface except in the special case when the Alfven and gas speeds are equal. In this case, the envelope proper is the "outer" part of the apple-shaped locus shown in Fig. 2b (the fast and slow loci merge analytically in this special case, producing the entire apple). The vertical dashed line (representing a disc) is a singular part

of the envelope. It is required by the general theory to make the locus convex, and it is also the correct limiting locus as $A \longrightarrow a$. However, this part of the characteristic locus is characterized by a two parameter conical family of rays; there is no unique ray which belongs to this surface element. Although this part of the locus is a carrier of discontinuities, it is not a carrier of the <u>dominant</u> discontinuity. For example, the image of an initial wave front across which first derivatives of (ρ, u, B) jump will have continuous first derivatives and will have a higher order discontinuity which propagates along these rays (Fig. 3).

The transverse locus is entirely singular. All plane elements pass through the two points $x = \pm A$ (the B-direction is taken as the x-axis). The "dominant" part of the locus consists of these two points, but the line joining them is also a singular envelope. This part of the locus is characterized by non-unique rays (a one parameter family). Using the transverse characteristic locus to obtain the motion of a wave front, one obtains the apparently one-dimensional undistorted propagation shown in Fig. 4. The wave front is forever contained within the flux tube in which it originated. The dotted part of the construction results fron the line joining $x = \pm A$ in the characteristic locus.

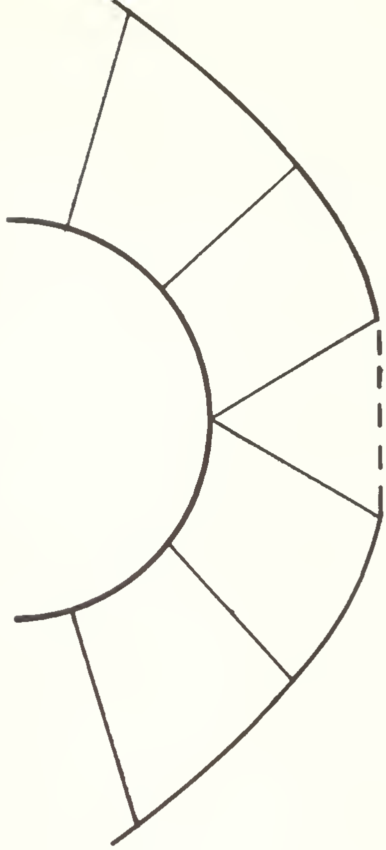

Figure 3

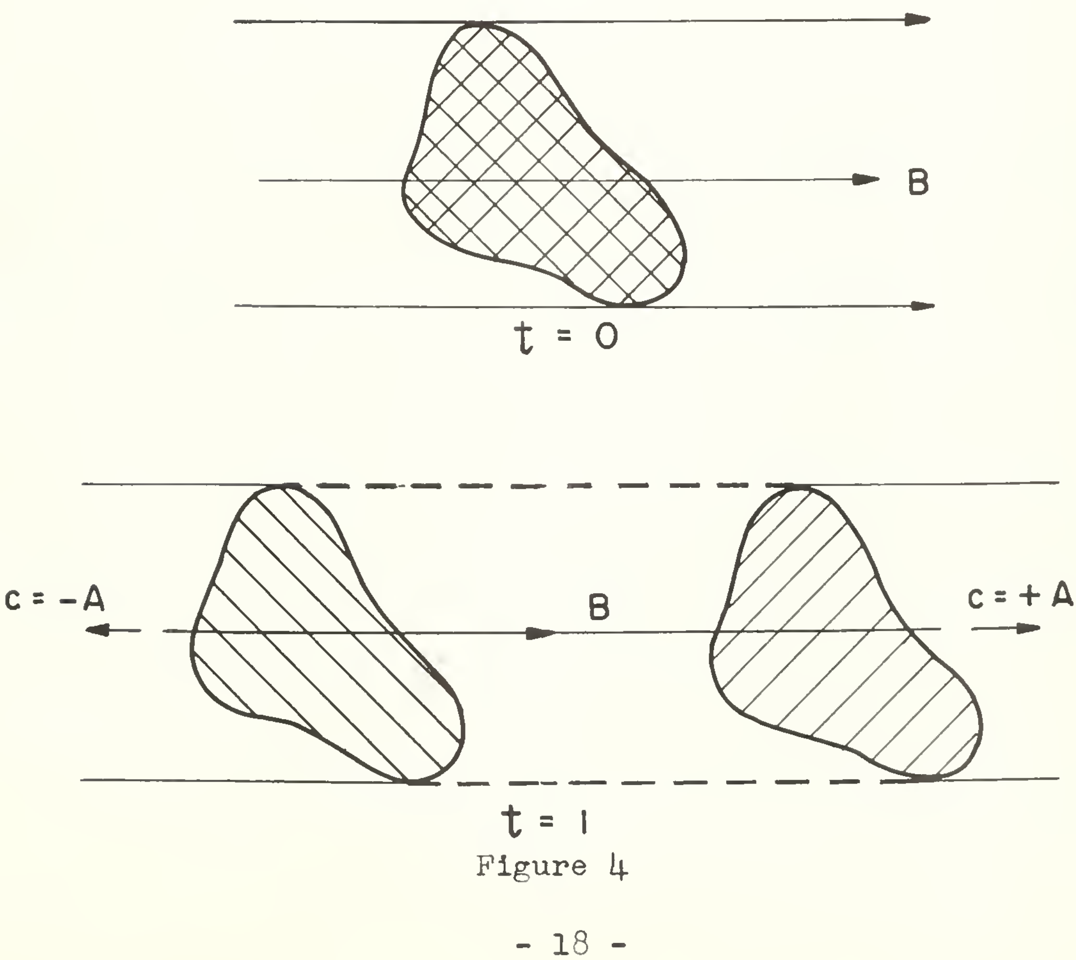

Figure 4

The regular part of the slow locus is not convex and has the two cusped lobes shown in Fig. 2. Again, there is a singular part of the envelope joining the two inner cusps. The distance of one of the outer cusps from the origin is greater than the maximum distance from the origin of the entire slow normal speed locus. In other words, a slow disturbance can propagate faster than the speed of any slow <u>plane</u> wave, and it can also propagate faster in some directions than a transverse disturbance.

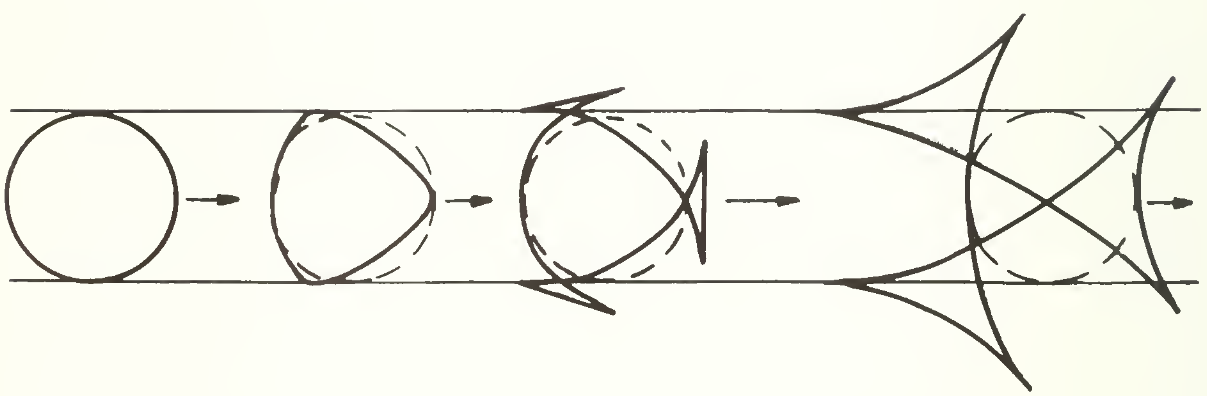

Figure 5

In Fig. 5 we show the evolution of an originally spherical slow wave front. For a certain time, the wave front is distorted but is not singular. During this time, the slow wave lags behind the transverse wave and remains within the original flux tube (consistent with the earlier construction based on the normal speed locus). However, cusps develop after a finite time, and the slow front can then overtake the transverse front. In particular, the front propagates three-dimensionally, penetrating outside the original flux tube.

The various portions of the characteristic locus represent possible carriers of discontinuities. With special initial data, part of the locus is dropped. We shall see later that the line joining the transverse points $x = \pm A$ is never present. It has not been verified whether the line joining the inner cusps of the slow locus is necessary. The complete answer would be given by evaluation of the Green's function for the hyperbolic system (2). The relevant parts of the characteristic locus would either carry singularities of the Green's function or bound regions of different analytical form.

4. One-Dimensional Problems

The characteristic form of the one-dimensional case of the nonlinear system (1) or of the linear system (2) is obtained by taking an appropriate linear combination of the equations to obtain a single equation in which all quantities, ρ, u, and B, are differentiated along the same direction in the (x, t) plane. This direction must be one of the characteristic directions. Six such characteristic equations can be obtained, one for each characteristic direction, to replace the six original equations. For the linear system with B_o constant, these characteristic differential equations can be integrated to yield six Riemann invariants, thus obtaining an explicit solution for the general initial value problem. For the nonlinear system, integration of the characteristic equations is possible only in special cases. This can be done if B is either parallel or perpendicular to the x-axis (x is the significant space coordinate in this one-dimensional problem). For B arbitrarily oriented, the system can be reduced to the integration of one ordinary differential equation in the special case of a simple wave; i.e. with initial data selected such that there is propagation along only one of the six characteristics.

We denote by u_n and B_n the x-components of u and B.

From div $B = 0$, we have $B_n =$ constant. We write u' and B' for the two-dimensional vector components perpendicular to the x-axis. We write the characteristic equations of (1) in terms of the differentiation operator,

(8) $$d = \frac{\partial}{\partial t} - (c - u_n)\frac{\partial}{\partial x}$$

If neither B_n nor B' vanishes, an elementary computation yields

(9) $$\begin{cases} B' \times (B_n du + c dB) = 0 \quad \text{on} \quad c = \pm A_n \\[2ex] \rho c^2 du_n + a^2 c\left(\frac{A_n^2}{c^2} - 1\right)d\rho - \frac{1}{\mu} B \cdot (B_n du + c dB) = 0 \quad \text{on } c = \pm c_s, \pm c_f \end{cases}$$

The transverse speed is given by $A_n^2 = B_n^2/\mu\rho$, and the slow and fast speeds are denoted by c_s and c_f.

If $B' = 0$ on some x-interval, we have $u' = 0$ and

(10) $$\rho du_n - c d\rho = 0 \quad , \quad c = \pm a \quad .$$

This is the ordinary gas dynamics result which can be integrated to yield the Riemann invariant

(11) $$u_n \mp \int \frac{a d\rho}{\rho} = \text{constant on} \quad c = \pm a \, .$$

If $B_n = 0$ on an interval, we have

$$
(12)\qquad \begin{cases} u' = \text{constant on } c = 0 \\ B'/\rho = \text{constant on } c = 0 \\ u_n \mp \int \frac{a^* d\rho}{\rho} = \text{constant on } c = \pm a^* \end{cases}
$$

where

$$
(13)\qquad (a^*)^2 = a^2 + B^2/\mu\rho \; ;
$$

a^* can be expressed as a function of ρ alone by using the integrated relation $B'/\rho = \text{constant}$ on $c = 0$ once the initial values of B and ρ are given.[1]

For the linear system, when B_n and B'_o are not zero, we obtain

$$
(14)\qquad \begin{aligned} & B'_o \times (B_n u + cB) = \text{constant on } c = \pm A_n \\ & \rho_o c^2 u_n + a_o^2 c \left(\frac{A_n^2}{c^2} - 1\right)\rho - \frac{1}{\mu} B_o \cdot (B_n u + cB) = \text{constant} \\ & \qquad \text{on } c = \pm c_s, \pm c_f \end{aligned}
$$

instead of (9). Also, in the case $B_n = 0$, we obtain the evident linearization of (12). However, in the case $B'_o = 0$, the linear result is entirely different, viz.

---

[1] This analogy with ordinary one-dimensional gas dynamics can be greatly extended. See A. A. Blank and Harold Grad, Fluid Dynamical Analogies, NYO-6486, Notes on Magneto-Hydrodynamics VII, Institute of Mathematical Sciences, New York University (July 15, 1958).

$$
(15) \quad \begin{cases} \rho_o u_n - c\rho = \text{constant on } c = \pm a_o \\ B_n u' + cB' = \text{constant on } c = \pm A_n . \end{cases}
$$

The second relation, on the characteristic $c = \pm A_n$, replaces the nonlinear result $u' = B' = 0$. We see that the restriction $B' = 0$ in the nonlinear case is much more stringent than the linearization about $B_o' = 0$ which allows a non-zero perturbation B'. The linear result (15) can be verified more rigorously by taking an appropriate limit, $B' \longrightarrow 0$, in the general nonlinear case (9). It is instructive to examine this limit for the linearized equations,

$$
(16) \quad \begin{cases} \dfrac{\partial \rho}{\partial t} + \rho_o \dfrac{\partial u_n}{\partial x} = 0 \\[2ex] \rho_o \dfrac{\partial u_n}{\partial t} + a_o^2 \dfrac{\partial \rho}{\partial x} = -\dfrac{1}{\mu} B_o' \cdot \dfrac{\partial B'}{\partial x} \\[2ex] \rho_o \dfrac{\partial u'}{\partial t} - B_n \dfrac{\partial B'}{\partial x} = 0 \\[2ex] \dfrac{\partial B'}{\partial t} - B_n \dfrac{\partial u'}{\partial x} = -B_o' \dfrac{\partial u_n}{\partial x} \end{cases}
$$

By ignoring the right sides of these equations, we obtain two decoupled wave equations, one for ρ and u_n with a_o as the propagation speed, and the other for u' and B' with A_n as the propagation speed. If B_o' is considered to be small, the right hand terms represent a small amount of coupling.

5. Transverse Waves[1]

We take the linear system with B_o constant, (2), and look for special solutions satisfying the following restrictions

$$(17)\qquad \begin{cases} \rho = 0 \\ \operatorname{div} u = 0 \\ u \cdot B_o = 0 \\ B \cdot B_o = 0 \end{cases}$$

Under these conditions, $\operatorname{curl} B \times B_o = (B_o \cdot \nabla)B$ and $\operatorname{curl}(u \times B_o) = (B_o \cdot \nabla)u$. Taking the direction of B_o as the x-axis, we obtain

$$(18)\qquad \begin{cases} \rho_o \dfrac{\partial u}{\partial t} = \dfrac{1}{\mu} |B_o| \dfrac{\partial B}{\partial x} \\[2ex] \dfrac{\partial B}{\partial t} = |B_o| \dfrac{\partial u}{\partial x} \end{cases}$$

This is a one-dimensional wave equation system representing the undistorted propagation of waves at a speed $A_o = B_o/\sqrt{\mu\rho_o}$ in the direction of B_o; y and z enter merely as parameters. The problem can be solved separately for each individual magnetic line in terms of the initial values on that line. In particular, any distrubance which is initially

---

[1] The special solutions given here were first presented in a series of lectures at General Electric in Philadelphia, July 1957.

found within a certain flux tube (of B_o) will forever remain within the same tube. For this special class of solutions, Fig. 4 (without the dotted lines) represents the propagation of the entire disturbance, not only the wave front. It follows by inspection that the solution of (18) satisfies (17) for all time provided that it does so initially. The term transverse is clearly appropriate for these waves since both u and B are perpendicular to the direction of propagation.

We have exhibited a class of special solutions of the system (2) which satisfy the special one-dimensional wave equation (18). Next we show that the two variables

$$(19) \qquad \begin{cases} j = B_o \cdot \text{curl } B \\ \omega = B_o \cdot \text{curl } u \end{cases}$$

satisfy the same system (18) if B and u are taken from an arbitrary solution of the full system (2). Performing the indicated operations on the last two equations of (2), we easily obtain

$$(20) \qquad \begin{cases} \rho_o \dfrac{\partial \omega}{\partial t} = \dfrac{1}{\mu} |B_o| \dfrac{\partial j}{\partial x} \\ \dfrac{\partial j}{\partial t} = |B_o| \dfrac{\partial \omega}{\partial x} \end{cases}$$

For any solution of the full system, j and ω propagate one-dimensionally at the apeed A_o .

Strangely enough, the special solutions which satisfy (18) are, in a certain sense, more than could have been expected from casual inspection of the original system (2). Since u and B in (18) are each two-component vectors, the characteristic "cone" (line, in this case) is counted twice; (18) is a fourth order system. The reason why this can occur will become clear later.

6. Compressive Waves

We again take the linear system (2) and this time look for equations which involve only the four quantities (17) which were taken to be zero in the last section. We write

$$(21)\qquad \begin{cases} \alpha = \operatorname{div} u \\ \sigma = \rho/\rho_o \\ \beta = B_x/|B_o| \\ A_o^2 = B_o^2/\mu\rho_o \end{cases}$$

and obtain, directly from the full set (2),

$$(22)\qquad \begin{cases} \dfrac{\partial\sigma}{\partial t} + \alpha = 0 \\[2ex] \dfrac{\partial \alpha}{\partial t} + a_o^2\Delta\sigma + A_o^2\Delta\beta = 0\ , \quad \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \\[2ex] \dfrac{\partial u_x}{\partial t} + a_o^2\dfrac{\partial\sigma}{\partial x} = 0 \\[2ex] \dfrac{\partial\beta}{\partial t} + \alpha - \dfrac{\partial u_x}{\partial x} = 0 \end{cases}$$

It is easily verified that this is a fourth order system and has as its two characteristic cones the slow and fast cones of the complete system (2). A particularly symmetric form is obtained if α and u_x are eliminated:

$$
(23) \quad \begin{cases} \dfrac{\partial^2\sigma}{\partial t^2} = a_o^2\left(\dfrac{\partial^2\sigma}{\partial x^2}+\dfrac{\partial^2\sigma}{\partial y^2}+\dfrac{\partial^2\sigma}{\partial z^2}\right)+A_o^2\left(\dfrac{\partial^2\beta}{\partial x^2}+\dfrac{\partial^2\beta}{\partial y^2}+\dfrac{\partial^2\beta}{\partial z^2}\right) \\[2ex] \dfrac{\partial^2\beta}{\partial t^2} = a_o^2\left(\dfrac{\partial^2\sigma}{\partial y^2}+\dfrac{\partial^2\sigma}{\partial z^2}\right)+A_o^2\left(\dfrac{\partial^2\beta}{\partial x^2}+\dfrac{\partial^2\beta}{\partial y^2}+\dfrac{\partial^2\beta}{\partial z^2}\right) \end{cases}
$$

Note that the second equation has a two-dimensional Laplacian operating on σ.

The adjective "compressive" will be used to refer to a combination of slow and fast waves even though we shall see later that there are special cases (indistinguishable from transverse waves) with $\operatorname{div} u = 0$.

The system (21) which is satisfied by the four variables $(\rho, \operatorname{div} u, u_x, B_x)$ taken from an arbitrary solution of the full system (2) is analogous to (20) for the variables $(B_o \cdot \operatorname{curl} B, B_o \cdot \operatorname{curl} u)$. Together, one can expect the combined system (19) and (21) to be equivalent to the original sixth order system; this will be verified in the next section. Special solutions of the full system which can be interpreted as compressive type (analogous to transverse type, (18)) are obtained as solutions of the fourth order system (21) by adjoining the conditions

$$
(24) \quad \begin{cases} j = B_o \cdot \operatorname{curl} B = 0 \\ \omega = B_o \cdot \operatorname{curl} u = 0 . \end{cases}
$$

Since j and ω satisfy the system (20), they are identically zero if they are taken zero initially.

7. <u>Resolution of an Arbitrary Wave into Components</u>

The results of the two previous sections can be summarized in the following table:

Variables	Transverse	Compressive
ρ, div u, u_x, B_x	zero	satisfy (22)
$(\text{curl } B)_x$, $(\text{curl } u)_x$	satisfy (20)	zero

Table I

The separation of an arbitrary disturbance into the two components transverse and compressive has been accomplished, but only by introducing new dependent variables. Given an arbitrary initial state (ρ, u, B), one can compute the values of the auxiliary variables $(\rho, \alpha, u_x, \beta, j, \omega)$ and then find the time variation of the two groups $(\rho, \alpha, u_x, \beta)$ and (j, ω) independently. We shall now attempt to describe this separation in terms of the original variables. This amounts to an inversion of the relations (19) and (21) to find (ρ, u, B) when given $(\rho, \alpha, u_x, B_x, j, \omega)$. This is nontrivial since it involves inversion of differential operators.

In terms of the auxiliary variables, the transverse component is defined by $\rho = \alpha = u_x = \beta = 0$ with j and ω given. The transverse component in the original variables

is given by $\rho = u_x = B_x = 0$ with u' and B' (the components perpendicular to B_o) to be found from

$$(25)\qquad \begin{cases} \mathrm{div}'\, u' = 0 \\ \mathrm{curl}'u' = \omega/|B_o| = \text{given} \end{cases} \qquad \begin{cases} \mathrm{div}'\, B' = 0 \\ \mathrm{curl}'B' = j/|B_o| = \text{given} \end{cases}$$

The accents (') denote two-dimensional operators in a plane x = constant. These differential equations determine u' and B' uniquely in each plane x = constant if a regularity condition (e.g. boundedness) is imposed at infinity. Exactly the same situation occurs for the compressive component except that the inhomogeneous terms occur in reverse:

$$(26)\qquad \begin{cases} \mathrm{div}'\, u' = \alpha - \partial u_x/\partial x = \text{given} \\ \mathrm{curl}'u' = 0 \end{cases} \qquad \begin{cases} \mathrm{div}'\, B' = -\partial B_x/\partial x = \text{given} \\ \mathrm{curl}'B' = 0 \end{cases}$$

Both the transverse and compressive components have now been uniquely determined, ρ, u_x, and B_x directly, and u' and B' by (25) and (26). Furthermore, the sums of the two components of u' and of B' are equal to the originally given values. For, the difference between u' and the sum of its two components is harmonic in each plane x = constant (i.e. both the operators div' and curl' annihilate it), and a vector which is harmonic in the whole plane and bounded at infinity vanishes identically.

We have obtained the resolution into components of the original variables. Unfortunately, this resolution seems to violate the domain of dependence properties which must hold for solutions of a hyperbolic system. For example, consider initial values (ρ, u, B) which are identically zero outside some bounded domain. The solution of the original system (2) will also be identically zero outside a certain bounded domain which grows with time. The two sets of variables (ρ, a, u_x, B_x) and (j, ω) behave similarly (but over different domains) since they also satisfy hyperbolic systems. However, the transverse or compressive components of u' and B' are obtained by solving (25) and (26) (essentially equivalent to Poisson's equation), and they will in general be different from zero over the entire plane x = constant, even though this is not true of the auxiliary variables. Of course, the sum of the transverse and compressive components of u and B must still be zero at large enough distances. In particular, the resolution of the initial values (ρ, u, B) has the property that the transverse and compressive components are equal and opposite outside the disturbed domain. We have the strange conclusion that the transverse and compressive components which initially cancel each other must continue to do so for some finite time until a wave from the initially disturbed

region catches up. Thus, these special initial values (viz. two-dimensional harmonic vectors in each plane x = constant) must propagate in exactly the same way as solutions of the transverse system which has a speed of propagation A_o in the x-direction and as solutions of the compressive system, with the complicated speeds of propagation described earlier. This is easily verified. As a matter of fact, it was noted in (17) and (18) that even more general sets of data (u' and B' not necessarily harmonic) propagate one-dimensionally at the speed A_o.

The fact that the resolution of initial values of u and B must be done in the large is a consequence of the property that a fast or slow wave front will, in general, leave behind it a transverse residue which then propagates one-dimensionally. The significant point of this analysis is that one can compute *a priori* what this residue will be; the transverse component of the *initial* u and B gives this residue on each magnetic line. In general, this residue will drop off as $1/(y^2 + z^2)$ for large distances.

One might be tempted to seek a resolution in a bounded domain by solving the equations (25) and (26) in such a domain and applying boundary conditions. However, in order for a set of initial values to be termed "transverse", it is necessary that the transverse jump conditions hold at the

boundary of the disturbed domain. For example, the condition $\operatorname{div} u = 0$ would have to be supplemented by $u_n = 0$, etc. One can extend Table I to obtain conditions which guarantee that a given bounded component is either transverse or compressive. However, it is not possible to expand arbitrary initial data as a sum of such modes. The essential point is that the difference between the given data and the sum of its transverse and compressive components is harmonic, but in a bounded domain, a harmonic vector need not be zero. As an example, consider initial data in a bounded domain with the property that the jumps at the boundary of the domain are transverse but the internal initial data are not. The "dominant" wave front will be a transverse one, but it will be preceded by a fast front which carries a higher order jump.

There is an alternative separation of the general solution of the system (2) into waves of transverse and compressive type which is, in a certain sense, simpler than the resolution into modes just described. We replace the system (2) by the compressive system (22) in the variables (a, ρ, u_x, B_x) together with the following exact equations for $\partial u'/\partial t$ and $\partial B'/\partial t$, taken directly from (2):

$$\rho_o \frac{\partial u'}{\partial t} - \frac{1}{\mu}(B_o \cdot \nabla) B' = \frac{1}{\mu} \operatorname{curl} \underline{B}_x x B_o - a_o^2 \nabla' \rho$$

(27)

$$\frac{\partial B'}{\partial t} - (B_o \cdot \nabla) u' = 0$$

The solution of a given initial value problem is obtained in two steps. First (a, ρ, u_x, B_x) are obtained from (22) using the given initial values. Then these variables are inserted into (27) which is solved for u' and B' using the given initial values of these variables. This is a one-dimensional wave equation for u' and B', but with a forcing term. The solution to (27) is the sum of a special solution which takes homogeneous initial values, $u' = B' = 0$, and a solution of the homogeneous equation with the given initial values. The homogeneous solution will have the one-dimensional domain of dependence of a transverse wave. The special solution will have the domain of dependence of a compressive wave since this is the region in which the forcing terms differ from zero.

The domain of dependence complications associated with the previous resolution into modes have disappeared. On the other hand, although the procedure just described gives a succinct description of the wave propagation features of the system under study, it does not offer a separation into modes. Specifically, the separation at $t = 0$ does not agree with the separation that would be obtained using the same procedure at a later time. To obtain the actual transverse and compressive components, one must split the initial values of u' and B' in the complex way described above rather than take them zero for one of the components. It is clear, from this analysis, that any splitting of initial values will leave unaltered the property of the sum that it is zero outside the compressive domain of dependence.

8. <u>The Limiting Cases</u> $A \gg a$ <u>and</u> $A \ll a$.

If the Alfven speed is either very large or very small compared to the gas sound speed, there is a further separation of solutions possible, viz. between the slow and the fast modes.

First we examine the normal speed and characteristic loci to obtain the limiting behavior of the <u>wave fronts</u> (which always separate, for any relation between A and a). From the characteristic determinant, (4), we find the slow and fast roots

$$(28) \qquad \begin{cases} A \gg a : & c^2 = A^2, \quad a^2\cos^2\theta \\ A \ll a : & c^2 = a^2, \quad A^2\cos^2\theta \end{cases}$$

where θ is the angle between B and the normal, n. In each of the two limiting cases, the fast normal speed locus is a sphere centered at the origin, and the slow locus consists of two spheres through the origin. From this, we conclude that the fast characteristic locus is spherical, representing isotropic three-dimensional propagation, whereas the slow locus consists of two points and the segment joining them, representing one-dimensional propagation in the direction B.

The complete description, including the transverse wave, is as follows. For $A \gg a$, there is an isotropic, three-dimensional spherical wave (fast) with A as the propagation

speed, and there are two one-dimensional waves, one (transverse) at the speed A and the other (slow) at the speed a. For $A \ll a$, there is a three-dimensional spherical wave (fast) at the speed a, and two superposed one-dimensional waves (slow and transverse) at the speed A.

A slightly finer examination shows that the cusped part of the slow characteristic locus shrinks to a point as A becomes either large or small compared to a. This means that it takes a relatively long time for the slow mode to deviate from one-dimensional propagation. Of course, the approximations made here cannot be expected to be valid uniformly for long times.

In order to investigate a possible separation between slow and fast wave motions (as distinguished from wave fronts), it is necessary to turn to the differential equations (22) for the compressive or slow plus fast component. Note that ρ and B_x have been made dimensionless using ρ_0 and B_0 as scale factors, (21), but u_x has been left dimensional since to choose either a_0 or A_0 as the scale factor might prejudge the result.

A complete separation of initial conditions can be obtained by Fourier transform, expanding in powers of a_0/A_0 or A_0/a_0. In the case $A_0 \gg a_0$, the relative magnitudes of σ, β, and u_x are independent of the wave number, k, to the lowest order in a_0/A_0; the separation of modes can

therefore be performed locally at each point in space. For $A_o \ll a_o$, the separation into modes involves k even to the lowest order in A_o/a_o; consequently the separation can be made only in the large. One must solve differential equations in order to separate the initial values into slow and fast components, just as was the case for some of the variables in the separation between transverse and compressive modes.

These results can be obtained more quickly by heuristic arguments stemming directly from inspection of the differential equations. Specifically, the fast mode is obtained by formally setting the slow sound speed equal to zero and the slow mode by letting the fast speed approach infinity. In essence, what we do is present the limiting differential equations whose solution amounts to inversion of the approximate Fourier transform obtained by taking a_o/A_o or A_o/a_o small.

Consider the case $A_o \gg a_o$. Setting $a_o = 0$ in (22) we obtain, for the fast mode

$$
(29) \qquad \left\{
\begin{aligned}
\frac{\partial \sigma}{\partial t} + \alpha &= 0 \\
\frac{\partial \alpha}{\partial t} + A_o^2 \Delta \beta &= 0 \\
\frac{\partial u_x}{\partial t} &= 0 \\
\frac{\partial \beta}{\partial t} + \alpha - \frac{\partial u_x}{\partial x} &= 0 \quad .
\end{aligned}
\right.
$$

We interpret $\partial u_x/\partial t = 0$ as $u_x = 0$ since this implies that the value of u_x propagates slowly (with zero speed, to this approximation). By the same argument, we conclude that $\beta = \sigma$ from $-\alpha = \partial\beta/\partial t = \partial\sigma/\partial t$. The system (29) can now be put in the form

$$(30) \quad \begin{cases} \dfrac{\partial^2\beta}{\partial t^2} - A_o^2 \Delta\beta = 0 \\ \sigma = \beta \\ u_x = 0 \\ \alpha = -\dfrac{\partial\beta}{\partial t} \end{cases} \qquad \begin{bmatrix} A_o \gg a_o \\ \text{fast mode} \end{bmatrix}$$

For the slow mode we let $A_o \longrightarrow \infty$ in (22) and obtain

$$(31) \quad \begin{cases} \dfrac{\partial\sigma}{\partial t} + \alpha = 0 \\ \beta = 0 \\ \dfrac{\partial u_x}{\partial t} + a_o^2 \dfrac{\partial\sigma}{\partial x} = 0 \\ \dfrac{\partial\beta}{\partial t} + \alpha - \dfrac{\partial u_x}{\partial x} = 0 \end{cases}$$

We interpret $\Delta\beta = 0$ as $\beta = 0$ in the infinite domain and obtain the simplified system

$$(32) \quad \begin{cases} \dfrac{\partial u_x}{\partial t} + a_o^2 \dfrac{\partial\sigma}{\partial x} = 0 \\ \dfrac{\partial\sigma}{\partial t} + \dfrac{\partial u_x}{\partial x} = 0 \\ \beta = 0 \\ \alpha = \dfrac{\partial u_x}{\partial x} \end{cases} \qquad \begin{bmatrix} A_o \gg a_o \\ \text{slow mode} \end{bmatrix}$$

We see that u_x is carried by the slow wave alone, and β is carried by the fast wave; σ and a are carried by both. By inspection of (30) and (32) we find the following resolution of initial values:

$$(33) \quad \begin{cases} u_x^f = 0 & u_x^s = u_x \\ \beta^f = \beta & \beta^s = 0 \qquad [A_o >> a_o] \\ \sigma^f = \beta & \sigma^s = \sigma - \beta \\ \dfrac{\partial \beta^f}{\partial t} = \dfrac{\partial u_x}{\partial x} - a & \end{cases}$$

The initial value of $\partial\beta^f/\partial t$ is required in addition to the value of β^f in order to solve the second order wave equation for β^f.

Summarizing, we see that the modes separate locally. The longitudinal component of the magnetic field perturbation is carried isotropically at the Alfven speed and the longitudinal component of velocity is carried one-dimensionally at the gas speed; the condensation σ, is carried by both waves. The slow wave is, in some sense, gas dynamical, but it is enormously modified by the presence of the magnetic field.

It is easily seen (cf. equation (26)) that the remaining components of vleocity and magnetic field, u' and B', are carried by the fast wave. In other words, the transverse residue is left by the fast wave alone.

Now we turn to the more difficult case, $A_o << a_o$. It is convenient to rewrite the system (22) in the following form:

$$(34) \quad \begin{cases} \dfrac{\partial \sigma}{\partial t} + \alpha = 0 \\ \dfrac{\partial^2 \sigma}{\partial t^2} - a_o^2 \Delta \sigma = A_o^2 \Delta \beta \\ \dfrac{\partial \phi}{\partial t} - \dfrac{\partial \psi}{\partial x} = 0 \\ \dfrac{\partial \psi}{\partial t} - A_o^2 \dfrac{\partial \phi}{\partial x} = A_o^2 \dfrac{\partial}{\partial x} \left(\dfrac{\partial^2 \sigma}{\partial y^2} + \dfrac{\partial^2 \sigma}{\partial z^2}\right) \end{cases}$$

where

$$(35) \quad \begin{cases} \phi = \Delta \beta - \left(\dfrac{\partial^2 \sigma}{\partial y^2} + \dfrac{\partial^2 \sigma}{\partial z^2}\right) \\ \psi = \Delta u_x - \dfrac{\partial a}{\partial x} \end{cases} .$$

It is convenient to consider the terms on the right in (34) as small coupling terms; these terms are removed when we examine the fast wave $(A_o = 0)$ or the slow wave $(a_o \longrightarrow \infty$ which implies $\Delta \sigma = 0$ or $\sigma = 0)$. To this approximation (and in these variables) the system splits into fast and slow modes:

$$(36) \quad \begin{cases} \dfrac{\partial \sigma}{\partial t} + \alpha = 0 \\ \dfrac{\partial^2 \sigma}{\partial t^2} - a_o^2 \Delta \sigma = 0 \\ \phi = \psi = 0 \end{cases} \qquad \left[\begin{array}{c} A_o << a_o \\ \\ \text{fast mode} \end{array}\right]$$

$$(37) \quad \begin{cases} \dfrac{\partial \phi}{\partial t} - \dfrac{\partial \psi}{\partial x} = 0 \\[2ex] \dfrac{\partial \psi}{\partial t} - A_o^2 \dfrac{\partial \phi}{\partial x} = 0 \\[2ex] \sigma = a = 0 \end{cases} \qquad \begin{bmatrix} A_o << a_o \\ \text{slow mode} \end{bmatrix}$$

The complication occurs (as in the separation of transverse from compressive waves - cf. section 7) when we return to the original variables via (35). For the fast mode, β and u_x are obtained by solving the Poisson equations

$$(38) \quad \begin{cases} \Delta\beta = \dfrac{\partial^2\sigma}{\partial y^2} + \dfrac{\partial^2\sigma}{\partial z^2} \\[2ex] \Delta u_x = \dfrac{\partial a}{\partial x} \end{cases} .$$

For the slow mode, β and u_x are obtained from

$$(39) \quad \begin{cases} \Delta\beta = \phi \\ \Delta u_x = \psi . \end{cases}$$

The given values of $\Delta\beta$ and Δu_x are split into a fast component as in (38) with the remainder as the slow component. The fast components of $\Delta\beta$ and Δu_x propagate three-dimensionally and the slow component, one-dimensionally.

It is not obvious (and is, as a matter of fact, true only to a certain approximation) that the sum of the separated modes cancels out sufficiently far from the disturbed

region. In order to verify this, it is convenient to consider an alternative separation into waves of approximately slow and fast type similar to the one described at the end of the last section. We solve the system (36) and (37) in two steps. First α and σ are found from (36). Then (37) is solved for the variables $\Delta\beta$ and Δu_x (rather than for ϕ and ψ) after inserting the known values of α and σ. The solution of this inhomogeneous system can be described as the sum of a homogeneous solution which takes the initial values (propagating one-dimensionally) and a special solution with zero initial values (propagating three-dimensionally as does the forcing term). This is <u>not</u> the separation into fast and slow modes. However, with this splitting, it is easy to see that Δu_x and $\Delta\beta$ are, for all time, the Laplacians of functions which vanish outside the domain of dependence. For the homogeneous solution, this property obtains initially and will continue to do so as the solution of a one-dimensional wave equation (cf. the argument in section 7). For the special solution, the conclusion follows provided that the inhomogeneous term is itself the Laplacian of a function which vanishes outside the domain of dependence. These inhomogeneous terms are

$$(40)\quad \begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial^2\sigma}{\partial y^2}+\frac{\partial^2\sigma}{\partial z^2}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\alpha}{\partial x}\right) = -\Delta\alpha \\ &\frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial x}\right)-A_o^2\frac{\partial}{\partial x}\left(\frac{\partial^2\sigma}{\partial y^2}+\frac{\partial^2\sigma}{\partial z^2}\right) = -\frac{\partial}{\partial x}\left\{a_o^2\Delta\sigma + A_o^2\left(\frac{\partial^2\sigma}{\partial y^2}+\frac{\partial^2\sigma}{\partial z^2}\right)\right\} \end{aligned}$$

To the approximation that $A_o << a_o$, the statement is verified. It is possible to take a slightly different approximation for the slow mode than (37), viz. the (exact) last two equations of (34); in this case the domain of dependence cancellation would hold exactly, but the separation of modes would only be approximate.

9. Applications

Perhaps the two most interesting features of the wave solutions discussed here are the complex wave front patterns which can arise, especially from the slow wave, and the fact that one-dimensional propagation can occur in a three-dimensional problem. In this section we discuss, very briefly, some possibilities of observing these effects.

In a liquid metal one will have $A_o << a_o$ with the attainable magnetic fields. The magnetic field and velocity perturbations carried by the fast wave are quite small, on the order of the condensation, $\sigma = \rho/\rho_o$. For the same reason, the slow residue left by the fast wave will be small. Consequently, one should expect to observe coincident slow and transverse modes, propagating as one-dimensional disturbances. The finite conductivity will exert a very large influence unless the apparatus is large and the magnetic field very strong. To observe the transverse or slow waves at all, the product of the magnetic field in gauss and the apparatus size in centimeters should exceed about 10^6 for mercury or 2×10^4 for liquid sodium. To detect the cusps developing from the slow wave is much more difficult; for this, the apparatus size is not so critical, but the magnetic field should be at least one million gauss in mercury or 10^5 gauss in sodium.

The one-dimensional propagation is of particular interest because it represents a transport of energy which is not

radially attenuated. Probably the most interesting example is the propagation of signals in the geomagnetic field. The theory presented here offers only a tentative indication of what may be expected. In addition to the poor conductivity and low degree of ionization, there are important effects associated with finite mean free path, finite gyro radius, and possibly finite Debye radius[1]. For most of the range of altitudes, the approximation $A_o >> a_o$ is valid. There exist two one-dimensional modes propagating along magnetic lines; one at the speed a_o and the other at the speed A_o. The question of general wave propagation into non-uniform magnetic fields B_o has not been treated here. However, the propagation of wave fronts has been completely analyzed[2] and is in this case strictly one-dimensional. There is, of course, a distortion of the wave front caused by the varying speed of propagation (see Figure 6). It is easy to compute that this distortion will not cause the amplitude of the signal to vary appreciably (i.e. by more than one order of magnitude) in propagating over a distance comparable to the earth's size.

A further question which must be answered is the transfer

---

[1] Some of these effects have been taken into account to a rough approximation in an analysis of "whistlers" by L. R. O. Storey, Trans. Roy. Soc., 246A, 113 (1953).

[2] See footnote on page 6.

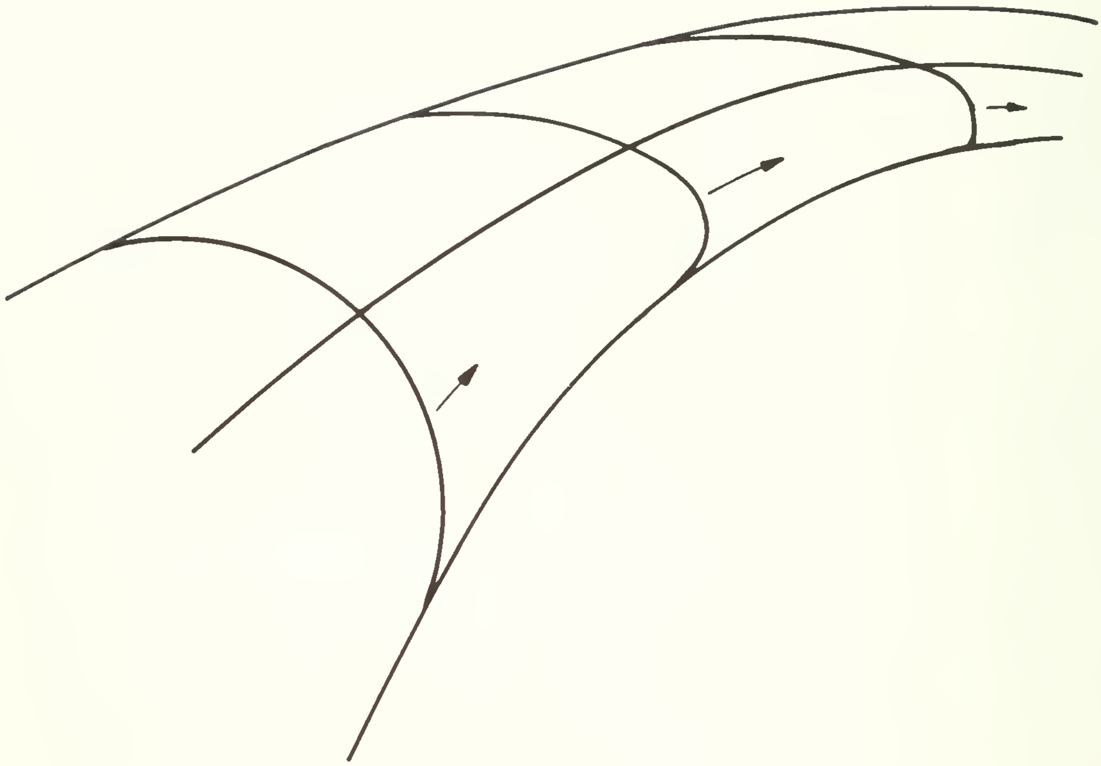

Figure 6

of these magnetohydrodynamic signals in the ionosphere through the atmosphere to the earth. The three modes existing in the conducting medium are coupled to two which exist in air; viz., ordinary sound waves and vacuum electromagnetic waves. The reflection and transmission coefficients across a discontinuity surface can be easily obtained (this has not yet been done), but the problem of a layer of variable properties will present more difficulty.

It is worth noting that the electromagnetic wave will be observed as a quasistatic magnetic field rather than as a conventional propagating wave. This is a consequence of the fact that the speed of light is much greater than the propagation speeds in the conducting medium. The electromagnetic disturbance in the atmosphere can be computed at each instant as a vacuum magnetic field which attains the instantaneous value of the normal component of B at the interface. The perturbation of B that is observed will be transverse and will be associated with the speed A.

In a laboratory plasma, the limitations of this theory are equally strong. However, it is at least worth remarking that the usual investigations which are limited to the propagation of plane waves only may very well mask unusual and important effects.

DATE DUE	BORROWER'S NAME	ROOM NUMBER
Fall'60	~~Reserve~~ (Grad)	
	~~T Cheng~~	
MAY 4 '65	S.P. 410	